SUSY INTERTWINING RELATIONS OF THIRD ORDER IN DERIVATIVES

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The general solution of the intertwining relations between a pair of Schrödinger Hamiltonians by the supercharges of third order in derivatives is obtained. The solution is expressed in terms of one arbitrary function. Some properties of the spectrum of the Hamiltonian are derived, and wave functions for three energy levels are constructed. This construction can be interpreted as addition of three new levels to the spectrum of partner potential: a ground state and a pair of levels between successive excited states. Possible types of factorization of the third order supercharges are analysed, the connection with earlier known results is discussed.

1. Introduction

During last two decades the supersymmetrical (SUSY) method [1] for investigation of different problems in nonrelativistic Quantum Mechanics (QM) occupied its stable position among other more conventional approaches. In particular, it seems to be most adequate for study the properties of isospectrality (or almost exact isospectrality) of pairs of quantum models and to construct such pairs ("quantum design"). Correspondingly, the experience

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of last years demonstrates that in the SUSY QM algebra the superinvariance of the Superhamiltonian:

$$[\hat{H}, \hat{Q}^{\pm}] = 0 \tag{1}$$

represents the main relation. Indeed, most of the known generalizations and applications of the standard Witten's SUSY QM [2] can be formulated as certain deformations of SUSY algebra, but Eq.(1), as a rule, remains unchanged (e.g. see [3], [4], [5]). In terms of the components of the Superhamiltonian $\hat{H}$ and supercharges $\hat{Q}^{\pm}$:

$$\hat{H} = \begin{pmatrix} \widetilde{H} & 0 \\ 0 & H \end{pmatrix}; \qquad \hat{Q}^{+} = (\hat{Q}^{-})^{\dagger} = \begin{pmatrix} 0 & 0 \\ Q^{-} & 0 \end{pmatrix}; \quad Q^{-} = (Q^{+})^{\dagger}, \tag{2}$$

Eq.(1) takes the form of SUSY intertwining relations:

$$\widetilde{H}Q^{+} = Q^{+}H; \quad Q^{-}\widetilde{H} = HQ^{-}. \tag{3}$$

Just these relations provide the isospectrality (up to possible zero modes of $Q^{\pm}$) of the partner Hamiltonians $\widetilde{H}, H$ and mutual connection between their wave functions:

$$\Psi_n(x) = Q^{-}\widetilde{\Psi}_n(x); \quad \widetilde{\Psi}_n(x) = Q^{+}\Psi_n(x). \tag{4}$$

The first extensions of **one-dimensional**[c] SUSY QM algebra, which preserve the intertwining relations (3), concerned the supercharges of higher orders in derivatives and were called as the Higher order SUSY Quantum Mechanics (HSUSY QM), or alternatively, N-fold SUSY QM, or Non-linear SUSY QM [3], [4], [7], [8], [9].

Until now the intertwining relations were completely investigated for the cases of first and second orders. In particular, two kinds of second order supercharges were proven [4] to exist: reducible supercharges (are equivalent to two successive first order supertransformations with real superpotentials) and irreducible supercharges (are not equivalent).

Much less attention was paid in the literature to the transformations of third order in derivatives (see the papers [10], [11], [12]). In [10] the particular form of third order

[c]On multi-dimensional extensions of Witten's SUSY QM see [6], [5] and references therein.

intertwining relations was investigated, which leads to a specific class of shape invariant potentials, while [12], [11] were mainly devoted to the construction of potentials with three lowest energy eigenvalues fixed. The main goal of this paper is to construct **the general solution** of intertwining relations (3) with supercharges of third order in derivatives.

The paper is organized as follows. The most general solution of the intertwining relations (3) with supercharges of third order in derivatives is derived in Section 2. From this general solution of intertwining relations some particular properties of the spectrum are obtained in Section 3, the explicit expressions for three wave functions with fixed energy eigenvalues are written. Thereby one of the partner Hamiltonians, H, can be considered as quasi-exactly-solvable [13], [11]. The alternative "quantum design" interpretation is also given here: starting from $\widetilde{H}$, involved in (3), one can build its partner H, whose spectrum includes three additional levels (a ground state ans a pair of levels between excited states). In Section 4 the variety of possible factorizations of the third order intertwining operators is investigated. In Section 5 some particular cases of the general solution are studied, and its classical limit is discussed.

2. The general solution of third order intertwining

We consider the intertwining relations (3) on the real axis with the most general supercharges of third order in derivatives:

$$Q^+ = (Q^-)^\dagger \equiv \sum_{n=0}^{3} f_n(x)\partial^n; \quad \partial \equiv \frac{d}{dx}; \quad x \in \mathcal{R} \tag{5}$$

where without any loss of generality $f_3(x) = 1$ is chosen. Thus one obtains a system of nonlinear differential equations for real functions $f_n(x),\ n = 0, 1, 2$ and real potentials $\widetilde{V}(x)$, $V(x)$:

$$\widetilde{V} - V = 2f_2', \tag{6}$$

$$f_2^2 - f_2' - 2f_1 - 3V = 3a, \tag{7}$$

$$2f_2' f_1 - f_1'' - 2f_0' - 3V'' - 2f_2 V' = 0, \tag{8}$$

$$f_0'' + V''' + f_2 V'' + f_1 V' - 2 f_0 f_2' \;=\; 0, \qquad (9)$$

where a is an arbitrary real constant, and $f' \equiv df/dx$. The derivative f_0'' can be found from (8), and after insertion into (9) one obtains:

$$(f_1 + V)''' - 2f_1(f_2' + V)' + 2f_2'(V - f_1)' + 4f_0 f_2' = 0. \qquad (10)$$

It is useful to introduce the function

$$W(x) \equiv V(x) + f_1(x) + a, \qquad (11)$$

in terms of which other unknown functions in (6) - (9) will be expressed. From (7) and (11) functions $f_1(x)$ and $V(x)$ can be written via $W(x)$ and $f_2(x)$:

$$f_1(x) \;=\; 3W(x) - f_2^2(x) + f_2'(x); \qquad (12)$$

$$V(x) \;=\; -2W(x) + f_2^2(x) - f_2'(x) - a. \qquad (13)$$

In order to express everything in terms of one function $W(x)$ it is useful to extract $f_0(x)$ from (8) and (10). Then one obtains the equation:

$$(W'' + 6W^2)' - 4W' f_2^2 - 4f_2' \int f_2 W' dx = 0,$$

and after its integration:

$$W'' + 6W^2 - 4f_2 \int f_2 W' dx = 2c, \quad c = \mathrm{const}. \qquad (14)$$

Multiplying (14) by W' and integrating again, one derives the relation between f_2 and W:

$$4\left(\int f_2 W' dx\right)^2 = W'^2 + 4(W^3 - cW + d); \quad d = \mathrm{const}, \qquad (15)$$

which gives the compact expression:

$$f_2 = \frac{W'' + 6W^2 - 2c}{4g}, \qquad (16)$$

where $g(x)$ is defined[d] by $W(x)$:

$$g(x) = \pm\frac{1}{2}\sqrt{W'^2 + 4(W^3 - cW + d)}. \tag{17}$$

So, both the coefficient functions $f_n(x)$ of (5) for $n = 0, 1, 2$ and potentials $\widetilde{V}(x)$, $V(x)$ are expressed in terms of an unique arbitrary function $W(x)$, thus giving the general solution of SUSY intertwining relations (3) of third order in derivatives by Eqs.(12), (13), (6), (8).

3. Spectrum and wave functions

Results of the previous Section lead to some consequences for the spectrum and wave functions of the Hamiltonian H. By direct (though rather cumbersome) calculations one can find that the operator Q^-Q^+ is represented as a third order polynomial of H with constant coefficients:

$$Q^-Q^+ = (H + a)^3 - (H + a)c + d. \tag{18}$$

Analogously, Q^+Q^- is a third order polynomial of $\widetilde{H}$ with the same coefficients, indicating that the anticommutator of supercharges (2) is a third order polynomial of the Superhamiltonian, i.e. one obtains the particular (third order) HSUSY QM [4] (see also [7] - [9]).

It follows from the positivity of Q^+Q^- Eq.(18) that all eigenvalues E_n of the Hamiltonian:

$$H\Psi_n(x) = E_n\Psi_n(x), \tag{19}$$

satisfy the inequality:

$$(E_n + a)^3 - (E_n + a)c + d \geq 0. \tag{20}$$

We will now restrict ourselves by consideration of wave functions (19) which are simultaneously the zero modes of the supercharge:

$$Q^+\Psi_k(x) = 0. \tag{21}$$

[d] We suppose that $g(x) \neq 0$.

We denote the energy eigenvalues, which correspond to these Ψ_k, by λ_k; $k = 0, 1, 2$. It is clear that λ_k are the roots of the third order polynomial:

$$(\lambda_k + a)^3 - (\lambda_k + a)c + d = 0. \tag{22}$$

We notice that all three roots of (22) $\lambda_0 \leq \lambda_1 \leq \lambda_2$ are real if the constant parameters d and c satisfy the inequalities:

$$c > 0; \quad 27d^2 \leq 4c^3. \tag{23}$$

In this case (20) implies that λ_0 necessarily coincides with the ground state[e]: $\lambda_0 \equiv E_0$, while λ_1, λ_2 coincide with two neighbouring excited states: $\lambda_1 \equiv E_j$; $\lambda_2 \equiv E_{j+1}$. Therefore all other (in general, an arbitrary number) energy eigenvalues E_n $(n \neq 0, j, j+1)$ may belong only to the following intervals:

$$E_n \in \{(E_0, E_j) \cup (E_{j+1}, \infty)\}. \tag{24}$$

Thus we deal with the so called quasi-exactly-solvable [13] system, i.e. with known positions of the ground state energy and two additional (not necessarily first excited) energy eigenvalues. In the case of only one real root λ_0 of (22), it realizes the ground state energy $\lambda_0 \equiv E_0$, and other eigenvalues (if they exist) are to belong to the semiaxis $E_n > E_0$.

These properties of the spectrum of H can also be interpreted from the "quantum design" point of view. Let us suppose that the spectrum and eigenfunctions of the **partner** Hamiltonian $\widetilde{H}$ are known. Then if $\widetilde{H}$ is involved in the intertwining relations (3), one can construct the Hamiltonian H with the same (due to (4)) energy values and **three** (in the case of three roots above) **additional** levels E_0, E_j, E_{j+1}, one of which E_0 becomes a ground state of H, and two other are inserted between a certain pair of **excited** states. In the case of **one** real root λ_0 only the ground state $\lambda_0 \equiv E_0$ can be included in the spectrum of H. Of course, saying about new states $\Psi_n(x)$; $n = 0, j, j+1$, we have in mind that their wave functions are normalizable. All eigenfunctions of these added states will be constructed below, and their possible normalizability will be investigated.

[e]We suppose that the corresponding wave function (see below) is normalizable.

By substitution of $\partial^2 \equiv (V(x) - H(x))$ the Eq.(21) for $k = 0, 1, 2$ is reduced to the linear first order differential equation for $\Psi_n(x)$; $n = 0, j, j+1$. Its general solution reads:

$$\Psi_n(x) = \exp\left(-\int \frac{V' + (V - E_n) f_2 + f_0}{V + f_1 - E_n} dx \right), \tag{25}$$

where E_n; $n = 0, j, j+1$ is an arbitrary of the roots of (22). By means of (7) - (9) the nominator in the integrand can be rewritten as:

$$V' + (V - E_n) f_2 + f_0 = (W - E_n - a) f_2 - W'/2 - g(x)$$

to express the wave functions in terms of function $W(x)$:

$$\Psi_n(x) = \mid W(x) - (E_n + a) \mid^{1/2} \exp \int \left(-f_2 + \frac{g(x)}{W - (E_n + a)} dx \right), \tag{26}$$

where $f_2(x)$ and $g(x)$ can be found from (16), (17). We remark that due to expression (26) one can keep the function $W(x)$ fixed and study dependence of the wave functions $\Psi_n(x)$ on their spectral parameters E_n. Vice versa, one can fix the position of energy level, studying the class of potentials (varying the form of function $W(x)$) with the same values of E_n.

It is appropriate to remind here that $W(x)$ in (25), (26) is an arbitrary function, which must provide for the normalizability of eigenfunctions $\Psi_n(x)$. To demonstrate that this restriction for $W(x)$ is not too strong, we give some examples of possible asymptotical behaviour of $W(x)$ at infinity and at finite singular point.

1) Let $W(x)$ has a growing power asymptotics at $x \to \pm\infty$. Then a negative valued $g(x)$ in (17) $g(x) \sim -W^{3/2}$ and $f_2 \sim \frac{3}{2} W^{1/2}$. In this case $\Psi(x)$ is normalizable if $\int W^{1/2} dx \to +\infty$ for $x \to \pm\infty$.

2) Let $W(x)$ has an asymptotical $x \to \pm\infty$ behaviour: $W(x) \to \gamma + \alpha x^{-2}$, where $\gamma(\gamma^2 - c) + d = 0$. Then $g(x) \sim \pm\sqrt{\alpha(3\gamma^2 - c)} x^{-1}$ and $f_2(x) \sim 2\sqrt{3\gamma^2 - c}\,\alpha^{-1} x$. Choosing a positive value for $g(x)$, one obtains normalizable $\Psi(x) \sim \exp(-\sqrt{3\gamma^2 - c}\,\alpha^{-1} x^2)$.

3) For $d = 0$ and $W(x) \to -\alpha x^{-2}$, $\alpha > 0$ at $x \to \pm\infty$ one has $g(x) \sim \pm\sqrt{c\alpha}\, x^{-1}$ and $f_2(x) \sim \mp\sqrt{c} x / 2\sqrt{\alpha}$. For the negative sign the wave functions are normalizable: $\Psi \sim \exp(-\sqrt{c} x^2 / 4\sqrt{\alpha})$.

4) If $W(x)$ has a singularity at $x=0$: $W(x)\to\alpha x^{-2}$, then $g(x)\sim\pm\alpha\sqrt{\alpha+1}x^{-3}$, and $f_2\sim\pm3\sqrt{\alpha+1}x^{-1}/2$. For the negative sign $\Psi\sim\mid x\mid^{\sqrt{\alpha+1}/2-1}$, i.e. $\Psi(x)$ is normalizable at $x=0$ for $\alpha>0$.

5) For a pole type singularity of $W(x)\sim\alpha x^{-1}$ at $x\to0$ one can take the negative sign of $g(x)\sim-\alpha x^{-2}/2$ and $f_2(x)\sim-x^{-1}$ to obtain normalizable $\Psi(x)$.

In all cases it is necessary to watch nodes of the function $g(x)$, which generate the singularities of $f_2(x)$, in order to keep the normalizability of wave functions under the control.

For illustration of the approach we consider a particular example with $W(x)=\gamma+\frac{\alpha}{1+x^2}$, where, according to item 2) above, we must choose $\alpha(3\gamma^2-c)>0$ and $d=\gamma(c-\gamma^2)$. Let us take in addition $\gamma\equiv-\alpha(\alpha+4)/12$ to ensure that the function $g(x)$ is an entire function:

$$g(x)=\frac{\alpha^2x(x^2+\frac{\alpha-2}{\alpha})}{2(1+x^2)^2};\qquad f_2(x)=\frac{\alpha x}{4}-\frac{\alpha-2}{4x}-\frac{2x}{1+x^2},$$

Then the partner potentials have (up to a common additive constant) the form:

$$V(x)\;=\;\frac{\alpha^2}{16}x^2+\frac{(\alpha-2)(\alpha-6)}{16}\frac{1}{x^2}; \tag{27}$$

$$\widetilde{V}(x)\;=\;\frac{\alpha^2}{16}x^2+\frac{(\alpha^2-4)}{16}\frac{1}{x^2}+\frac{4}{1+x^2}-\frac{8}{(1+x^2)^2}+\frac{\alpha}{2}. \tag{28}$$

The singular harmonic oscillator potential (27) on the entire axis[f] was studied in the literature (e.g. see [14]). We would not like to make here the detailed study of this potential our aim, pointing out the paper [15], where this kind of singular potentials was investigated just in the frameworks of SUSY QM and unbroken shape invariance. Summarizing [15], the spectrum for the transition (see for definitions [16]) "soft" (i.e. for $-\frac{1}{4}<\frac{(\alpha-2)(\alpha-6)}{16}<\frac{3}{4}$) potential (27) was shown to consist of two equidistant sequences of harmonic-oscillator-like levels, and corresponding wave functions were also constructed.

The condition of "softness" of both transition potentials (27) and (28) means that $\alpha\in(0,4)$. Thus, the third order intertwining of $V(x)$ and $\widetilde{V}(x)$ ensures that the spectrum and

[f] Though over this paper we deal with the problems on a whole line, it can be directly restricted to a half axis $x>0$ by imposing suitable conditions for behaviour of wave functions at the origin.

wave functions[g] of the potential (28) coincide (as usual, up to zero modes of Q^+) with those of the partner potential (27).

4. The factorization of intertwining operators

The important question concerns the possible factorization of the general intertwining operator (5) onto the lower order multipliers. We start from the factorization onto three operators:

$$Q^+ \equiv -q_1^+ q_2^+ q_3^+, \tag{29}$$

where $q_i^+;\ i = 1, 2, 3$ are of the first order in derivatives[h]:

$$q_i^+ \equiv -\partial + W_i(x). \tag{30}$$

Eq.(29) leads to the following system of nonlinear differential equations between superpotentials $W_i(x)$ and coefficient functions $f_n(x)$:

$$f_2 = -(W_1 + W_2 + W_3); \tag{31}$$

$$f_1 = -(W_2 + W_3)' + W_1(W_2 + W_3) + W_2W_3 - W_3'; \tag{32}$$

$$f_0 = (W_2W_3)' - W_1W_2W_3 - W_3'' + W_1W_3'. \tag{33}$$

One can take $W_2 + W_3$ from Eq.(31), insert it into Eq.(32) and use the result in Eq.(33). In such a way one obtains the nonlinear second order differential equation for $W_1(x)$:

$$(W_1^2 - W_1')' - (W_1^2 - W_1')(W_1 + f_2) + W_1(2f_2' - f_1) - f_2'' + f_1' - f_0 = 0. \tag{34}$$

Eq.(34) can be linearized by defining $W_1(x)$ as a logarithmic derivative of a function $y(x)$:

$$y''' - f_2y'' + (f_1 - 2f_2')y' + (f_1' - f_2'' - f_0)y = 0; \quad W_1(x) \equiv -\frac{y'(x)}{y(x)}. \tag{35}$$

[g] The nonsingular ($\alpha = 2$) particular case of (28) was discussed in [11].

[h] We choose the signs in operators q^+ the same as in the conventional first order SUSY QM [1].

Thus an arbitrary of solutions of (35) gives the real superpotential W_1 in terms of real functions $f_n(x)$. Then one can use (32), (33) to find also W_2 and W_3 in terms of f_n.

It is necessary to remark that for the real valued solution W_1 one can consider **two different opportunities** for W_2 and W_3 :

1) real valued W_2, W_3; and

2) mutually conjugated complex $W_2 = W_3^\star$ with **constant** imaginary part[i].

The first option can be refered to as realizing a maximally reducible supercharge (supertransformation). The first order multipliers q_i^+ create a chain ("dressing chain" [17]) of Hamiltonians $(-\partial^2 + (W_i^2 \mp W' + \epsilon_i)$, such that the neighbouring Hamiltonians (up to a constant) in the chain are intertwined by q_i^+. This procedure is a particular case of the familiar construction of reducible Higher order SUSY QM [3], [4], [11] by gluing. In this case the polynomial (22) has three real roots E_n, $n = 0, 1, 2$.

The second option corresponds to partially reducible supercharges, where $Q^+ = q_1^+ M_{23}^-$ with irreducible second order operator with real coefficient functions:

$$M_{23}^- \equiv q_2^+ q_3^+ = (-\partial + W_2(x))(-\partial + W_2^\star(x)). \tag{36}$$

Though the superpotentials W_2, W_3 are chosen here to be complex, the coefficient functions f_n and potentials $\widetilde{V}, V$ still are kept real[j]. Just this kind of intertwining operators was investigated in [3], [4]. In this case (22) has one real and two mutually conjugated complex roots λ_k.

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[i] In this case the third order polynomial (22) has one real and two mutually conjugated roots.

[j] The complex valued f_n (with real $\widetilde{V}, V$) would lead to additional intertwining relations of lower order (see for details the preprint in [10] and [9].

5. Some particular cases of intertwining

5.1. Quasi-exactly-solvable models with three lowest levels given

The particular case of the first (maximally reducible) option above has to be compared with the construction of the paper [11], where the intertwining operators of third order were useed to build the quasi-exactly-solvable Hamiltonians, with three **lowest** values of bound state energy given. In [11] from the very beginning the intertwining operator had the factorized form with real superpotentials $W_i(x)$. Those superpotentials were generated by the **groung state** wave functions $\Psi_0^{(i)}(x)$ of three consecutive Hamiltonians $H^{(i)}$; $i = 1,2,3$, included in the chain [4] of three supersymmetric transformations of first order with intermediate gluing (up to a constant shift):

$$H^{(i)}\Psi_0^{(i)}(x) = E_0^{(i)}\Psi_0^{(i)}(x); \quad \Psi_0^{(i)}(x) \equiv \exp\Bigl(-\int W_i(x)dx\Bigr). \tag{37}$$

It was shown in [11] that all three superpotentials $W_i(x)$ and corresponding potentials can be expressed in terms of one function $U(x)$ with definite strong limitations in its analytical properties originated from the restrictions on logarithmic derivative of normalizable ground state wave functions (37): $\int W_i(x)dx$ has no singularities, and asymptotically $\mathrm{sign} W_i(x) = \pm 1$ at $x \to \pm\infty$. It can be shown by rather long calculations that the function $U(x)$ of [11] and $W(x)$ in Section 2 of the present paper coincide up to the constant if the condition (23) is fulfilled. Thus the solutions described in [11] can be obtained as particular cases of the general construction presented above, if one will restrict himself by suitable strong conditions for analytical and topological properties of functions and for relations between constants. In the general construction of the present paper both the singularities and the complexity of functions W_i are allowed if they preserve potentials $\widetilde{V}, V$ to be real and not too singular.

We remark that each of two kinds of factorization of Q^+ depends on one arbitrary (up to some discrete restrictions) function. It is obvious that the alternative to option 2) partially reducible factorization of Q^+ exists, if one chooses W_3 real and $W_1 = W_2^\star$ with constant imaginary part. The third similar opportunity with W_2 real and $W_1 = W_3^\star$ is equivalent to

the previous ones due to commutation $[q_1^+, q_2^+] = [q_3^+, q_2^+] = 0$.

5.2. Models with shape invariance of third order

We will now consider the particular case of intertwining relations (3) of third order where $\widetilde{H}$ and H **coincide up to a constant** 2λ :

$$HQ^+ = Q^+(H + 2\lambda). \tag{38}$$

This (shape invariant) sort of intertwining appeared in [10] in a natural way when a pair of Hamiltonians $\widetilde{H}, H$ participated simultaneously in two intertwining relations - of first (by operators q^+) and second (by operators M^-) order in derivatives.

$$q^+ = -\partial + \omega(x)); \quad M^- = \partial^2 + 2\phi(x)\partial + b(x) + 2\phi'(x)), \tag{39}$$

where (see [10], [4] for details)

$$b(x) = -\phi'(x) + \phi^2(x) - \frac{\phi''(x)}{2\phi(x)} + \frac{\phi'^2(x)}{4\phi^2(x)} + \frac{\beta}{4\phi^2(x)}; \quad \beta > 0; \tag{40}$$

$$\phi''(x) = \frac{(\phi'(x))^2}{2\phi(x)} + 6\phi^3(x) + 8\lambda x\phi^2(x) + 2(\lambda^2x^2 - (\lambda + \alpha))\phi(x) + \frac{\beta}{2\phi(x)} \tag{41}$$

The specific shape invariance [18] property of the Hamiltonian H, realized by (38), allowed to find its spectrum algebraically [10] by analysis of zero modes of Q^+ and Q^- :

$$Q^+ \equiv q^+M^-; \quad Q^- = M^+q^-. \tag{42}$$

Comparing coefficients of ∂^k in (42), one obtains the relations:

$$f_2(x) = 2\phi(x) - \omega(x);$$
$$f_1(x) = 4\phi'(x) - 2\omega(x)\phi(x) + b(x). \tag{43}$$

The function $f_2(x)$ is fixed by (6) and (38):

$$f_2(x) = -\lambda x; \quad \omega(x) = 2\phi(x) + \lambda x.$$

It follows from Eqs.(12), (14) and (43) that

$$W(x) = \phi'(x) - 2\phi^2(x) - 2\lambda x\phi(x) + \frac{2\lambda + \alpha}{3}$$

and, due to (14), has to satisfy nonlinear differential equation:

$$\left(\frac{W''(x) + 6W^2(x) - 2c}{x}\right)' = 4\lambda^2 x W'(x). \tag{44}$$

Indeed, it can be shown by direct calculation (using (41)) that (44) is satisfied for $c = \frac{(2\lambda+\alpha)^2}{3} - \beta$. Thus the shape invariant case (38) is included in the general solution of Section 2.

5.3. Models with intertwining of first and third orders

The third order intertwining of the previous Subsection was based on the **double intertwining** of the same pair of Hamiltonians $\widetilde{H}, H$ - by operators q^+ of the first and M^- of the second orders. Now we will consider the double intertwining by operators q^+ of first order and Q^+ of third order in derivatives. The first intertwining gives:

$$\tilde{H} = q^+q^- = -\partial^2 + \omega^2(x) - \omega'(x) + \epsilon; \quad H = -\partial^2 + \omega^2(x) + \omega'(x) + \epsilon; \;\; \epsilon = \text{const},$$

while the second one (see (6), (13)):

$$V(x) = -2W(x) + f_2^2(x) - f_2'(x) - a = \widetilde{V}(x) - 2f_2'(x).$$

Thus,

$$\begin{aligned} \omega(x) &= -f_2(x) - \gamma; \;\; \gamma = \text{const}; \\ 2\gamma f_2(x) &= -2W(x) - (a + \epsilon + \gamma^2). \end{aligned} \tag{45}$$

Two cases have to be considered separately: $\gamma = 0$ and $\gamma \neq 0$.

For $\gamma = 0$ from (45) $W(x) = -\frac{1}{2}(a + \epsilon) \equiv \kappa = \text{const}$. Then from Eqs.(6) - (10) one can derive that

$$\begin{aligned} f_1(x) &= 3\kappa - f_2^2(x) + f_2'(x); \\ f_0(x) &= 3\kappa f_2(x) - f_2^3(x) + f_2''(x) - f_2(x)f_2'(x); \\ V(x) &= f_2^2(x) - f_2'(x) - 2\kappa - a. \end{aligned}$$

Then the operator Q^+ is reduced ("modulo Hamiltonian H") to the **first order** operator

$$Q^+ = (\kappa - a - H)q^+.$$

More interesting case is for $\gamma \neq 0$, when

$$f_2(x) = -\gamma^{-1}W(x) - \nu; \quad \nu = \text{const.}$$

Substitution of this expression into Eq.(15) gives equation, which includes the function $W(x)$ only:

$$(W')^2 = \gamma^{-2}W^4 + 4(\nu\gamma^{-1} - 1)W^3 + 4\nu^2W^2 + 4cW - 4d. \tag{46}$$

In general case its solution can be expressed in terms of elliptic functions, but some particular values of constants (e.g. $\nu = c = d = 0$) in (46) give simpler solutions as well.

It can be shown that now the operator Q^+ is expressed (again "modulo Hamiltonian H") in terms of **first order** and **second order** operators

$$Q^+ = L^+ - Hq^+; \quad L^+ \equiv -\gamma^{-1}\partial^2 + (W - a)\partial - V\omega + \omega''.$$

Therefore the same pair of Hamiltonians is intertwined both by the first order q^+ and the second order L^+ operators. This result is in accordance with the general statements of the recent paper [9] concerning "the optimal set of two basic SUSY generators of even and odd order". One can check also from the intertwining relations that the third order operator $R \equiv (L^+)^\dagger q^+$ commutes with H, playing the role of its symmetry operator (see discussion in the preprint version of [10] and [9]).

5.4. The classical limit

It is known that sometimes results obtained for the quantum (supersymmetrical) systems can be successfully used to investigate its classical prototype (e.g. see [19] on two-dimensional second order SUSY Classical Mechanics). We will consider here the consequences of third order SUSY QM for the classical limit $\hbar \to 0$. With this object we have to restore the $\hbar$ in

all formulas of Section 2. It can be done by rewriting differential operators in terms of the momentum $p = -i\hbar\partial$. The mnemonic rule is to multiply each derivative by the multiplier $\hbar$ in Eqs.(5) - (10). Therefore, in the classical limit $\hbar \to 0$ one obtains from (6) - (9):

$$\begin{aligned} \widetilde{V} - V &= 0; \\ (f_2^2 - 2f_1 - 3V)' &= 0; \\ f_2' f_1 - f_0' - f_2 V' &= 0; \\ f_1 V' - 2f_0 f_2' &= 0. \end{aligned} \tag{47}$$

The general solution of this "classical system" of equations can be obtained from its quantum counterpart of Section 2 by mnemonic rules formulated above. It is useful to compare this problem with discussion in the paper [8], where the function $B(x, ip) = \sum b_{3-n}(ip)^n$ plays the role of classical analogue of the quantum operator Q^+ with $b_{3-n}(x) \equiv f_n(x)$. These coefficient functions have to satisfy the differential equation (Eq.(2.24) of [8]):

$$b_n'(x) + (5-n)b_{n-2}(x)V'(x) - L(x)b_{n-1}(x) = 0, \tag{48}$$

which was not solved in the framework of pure classical theory (see details in [8]). However Eq.(48) coincides exactly (up to some normalization factors) with Eq.(47), whose **general solution** can be found analytically from the quantum solution of Section 2 by $\hbar \to 0$ limit.

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References

[1] G. Junker, Supersymmetric Methods in Quantum and Statistical Physics, Springer, Berlin, 1996;
F. Cooper, A. Khare, U. Sukhatme, Phys.Rep. 25 (1995) 268;
B.K. Bagchi, Supersymmetry in Quantum and Classical Mechanics, Chapman and Hall/CRC, Boca Raton, 2001.

[2] E. Witten, Nucl.Phys. B 185 (1981) 513.

[3] A.A. Andrianov, M.V. Ioffe, V.P. Spiridonov, Phys.Lett. A 174 (1993) 273.

[4] A.A. Andrianov, M.V. Ioffe, F. Cannata, J.-P. Dedonder, Int.J.Mod.Phys. A 10 (1995) 2683.

[5] A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze Phys.Lett. A 201 (1995) 103;
A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, Theor.Math.Phys. 104 (1995) 1129;
A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, solv-int/9605007; Published in: Zapiski Nauch. Seminarov POMI RAN (ed.L.D.Faddeev et.al.), 224 (1995) 68 (In Russian); Translation in: Problems in QFT and Statistical Physics (ed.L.D.Faddeev et.al.) 13;
A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, J.Phys.:Math.Gen. A 32 (1999) 4641;
F. Cannata, M.V. Ioffe, D.N. Nishnianidze, J.Phys.:Math.Gen. A 35 (2002) 1389;
A.A. Andrianov, F. Cannata, M.V. Ioffe, D.N. Nishnianidze, J.Phys.:Math.Gen. A 30 (1997) 5037.

[6] A.A. Andrianov, N.V. Borisov, M.V. Ioffe, Phys.Lett. A 105 (1984) 19;
A.A.Andrianov, N.V. Borisov, M.V. Ioffe, M.I. Eides, Phys.Lett. A 109 (1985) 143.

[7] B.F. Samsonov, Mod.Phys.Lett. A 11 (1996) 1563;
V.G. Bagrov, B.F. Samsonov, Phys.Part.Nucl. 28(4) (1997) 374;
D.J. Fernandez C., M.L. Glasser, L.M. Nieto, Phys.Lett. A 240 (1998) 15;
S. Klishevich, M. Plyushchay, Mod.Phys.Lett. A 14 (1999) 2739;

D.J. Fernandez C., J. Negro, L.M. Nieto, Phys.Lett. A 275 (2000) 338;
M. Plyushchay, Int.J.Mod.Phys. A 15 (2000) 3679;
H. Aoyama, M. Sato, T. Tanaka, Phys.Lett. B 503 (2001) 423;
H. Aoyama, M. Sato, T. Tanaka, Nucl.Phys. B 619 (2001) 105;
R. Sasaki, K. Takasaki, J.Phys.:Math.Gen. A 34 (2001) 9533.

[8] S. Klishevich, M. Plyushchay, Nucl.Phys. B 606[PM] (2001) 583.

[9] A.A. Andrianov, A.V. Sokolov, Nucl.Phys. B 660 (2003) 25.

[10] A.A. Andrianov, F.Cannata, M.V. Ioffe, D.N. Nishnianidze, Phys.Lett. A 266 (2000) 341; preprint quant-ph/9902057.

[11] T.V. Kuliy, V.M. Tkachuk, J.Phys.:Math.Gen. A 32 (1999) 2157.

[12] N. Debergh, J. Ndimubandi, B. Van den Bossche, quant-ph/0209080.

[13] A.V. Turbiner, Comm.Math.Phys. 118 (1988) 467;
A.G. Ushveridze, Sov.J.Part.Nucl. 20 (1989) 504.

[14] L. Lathouwers, J.Math.Phys. 16 (1975) 1393;
A. Jevicki, J. Rodrigues, Phys.Lett. B 146 (1984) 55;
A. Das, S.A. Pernice, Nucl.Phys. B 56 (1999) 357.

[15] R. Dutt, A. Gangopadhyaya, C. Rasinariu, U. Sukhatme, J.Phys.:Math.Gen. A 34 (2001) 4129.

[16] W.M. Frank, D.J. Land, Rev.Mod.Phys. 43 (1971) 36.

[17] A.P. Veselov, A.B. Shabat, Funct.Anal.Appl. 27 (1993) 81.

[18] L.E. Gendenshtein, JETP Lett. 38 (1983) 356.

[19] A.A. Andrianov, M.V. Ioffe, D.N. Nishnianidze, J.Phys.:Math.Gen. A 32 (1999) 4641.